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THE

ASSURANCE MAGAZINE.

On the Doctrine of Successive Lives. By PETER GRAY, ESQ. F.R.A.S.

INTRODUCTION.

THE earlier writers on the subject of Successive Lives were singularly unfortunate in their investigations. One and all, including De Moivre, Dodson, Simpson, Morgan, and Baily, were guilty of an oversight by which their results were vitiated and rendered unfit for use. (On this point see Milne, note ix., pp. 708–710, and De Morgan on 'Probabilities,' Appendix the Second, pp. xv.–xviii.) The leading problems connected with this subject were first correctly solved by Mr. Milne, who, however, presented his results in a form exceedingly ill-adapted for numerical application. (See his work, pp. 240–250.) And in his singular mode of proceeding he has been followed by Mr. Jones (pp. 186–189). Mr. Griffith Davies, in his uncompleted work on Assurances, enters on what must be considered the right track, and, so far as he goes, presents his results in an unobjectionable form. Unfortunately the work breaks off in the very middle of the investigations on this subject, and many cases are consequently left unprovided for.* Mr. De Morgan (*Essay on Probabilities,* pp. 228–236) enters on the subject so far as the plan of his work permits, and gives rules for the leading cases, which are of course correct. In his Second Appendix also (pp. xv.–xxii.) he demonstrates those rules, and shows (as already intimated) wherein the rules of the earlier writers are defective.

The *Philosophical Magazine* for January 1850 contains a paper by Mr. Weddle, in which the subject is treated with that gentleman's usual ability and elegance; and the present writer is free to confess that had Mr. Weddle's paper been previously known to him he should not have thought it necessary to prepare his own. But the latter was prepared, and fully written out in its present form as far as the end of section 68, and with hardly the change of a word, at least six years ago; and on now going over it, he finds that the method of investigation is so different from Mr. Weddle's (although the results, so far as these are common to both investigations, are the same), and the paper generally—owing perhaps to its more elementary character—is so much fuller in its details than that of Mr. Weddle, that he ventures to think even the present late publication of it will not be unacceptable. He would direct attention in particular to Problem III., which, although so all-important on account of its practical applications that without it the subject cannot be considered complete, has been altogether overlooked by Mr. Weddle, as well as by every preceding writer.

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* The present author thinks it right to state that the following paper was prepared before he had an opportunity of seeing Mr. Davies's work.

In conclusion, the author has only further to allude to the very peculiar character of the resulting formulæ. They do not, like the formulæ which arise in other inquiries connected with the subject of life contingencies, consist of *sums* of products of tabulated quantities by what may be called variable quantities; but they consist in almost every case of *a single continued product* of quantities, all of which are, or may very easily be, tabulated once for all. They are hence singularly adapted to logarithmic computation.

London, 26 *June*, 1851.

PART I.—THEORY.

(1.) BY the term *status*, as used in connection with this subject, is meant "the state or condition of things, during the continuance of which an annuity is to be paid;"* or, on the cessation or dissolution of which a sum assured is to be paid.

(2.) Thus a status may be the duration of a specified term of years, the duration of a single life, the joint duration of two or more lives, the duration of the longest survivor of any given number of lives; or, in fine, either the joint duration or the longest duration of any two or more of the foregoing or similar "states of things."

(3.) By the *value* of a status is meant the value of an annuity of £1 upon that status; that is, of £1 per annum to be paid so long as that status should subsist.

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* De Morgan on Probabilities (Lardner's Cyclopædia), p. 190.

(4.) One status may succeed another; and this succession may be of two kinds. The status in expectation (as that which is to succeed may be termed) may be either a status now in existence, which, on the failure of that at present in possession, is to take its place in the enjoyment of the benefit, whatever this may be; or, it may be one that is not to be nominated till the failure of that in whose place it is to be put.

(5.) Thus, in the case of an annuity which at the demise of the present possessor is to be entered upon and enjoyed by another individual *now known and specified,* we have an example of a succession of the first kind. But if all that is now determined with regard to the succeeding life be, that at the period of his entering on possession he should be of a specified age, the succession is of the second kind.

(6.) It thus appears that a marked distinction between the two kinds of succession is this: in the first, the succession *may* never take place, since the status in expectation may fail before that in possession; in the second, the succession *must* take place, since the succeeding status is not selected till after the failure of that in possession, that is, until the place in which it is to be put is unoccupied.*

(7.) It is to successions of the second kind that we purpose in the following pages to confine our remarks, those of the first kind having been already very ably and satisfactorily treated.

(8.) It may be proper, before going farther, to remind the reader that an annuity upon any status, that is payable yearly (as is always to be understood as meant, unless the contrary is expressed), makes its payments at the end of each year during the continuance of that status. Hence, if it be not described as *due,* its first payment is made on the first anniversary of its being entered upon, and its last on the last anniversary of the same event which occurs during the subsistence of the status on which it depends. But if the annuity be described as *due,* it makes one payment also at the time of its being entered upon; so that, at that period, an annuity due is worth just that one payment more than an annuity that is not so described.

(9.) The same things hold, *mutatis mutandis,* if the annuity be payable at other than yearly intervals; for example, half-yearly or quarterly. That is to say, the successive payments are made at the end of the respective intervals during the subsistence of the status; and if the annuity be due, a payment is also made at the beginning of the first interval, that is, when the annuity is entered upon.

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* De Morgan on Probabilities, p. 228.

(10.) An assurance upon any status makes its payment, unless the contrary be expressed, at the end of the year in which that status fails; that is, just a year after the last payment of an annuity on the same status.

(11.) And in regard to both an annuity and an assurance, if no sum is mentioned, £1 is to be understood as the annual rent of the first and the amount of the second.

(12.) When the value of an annuity or an assurance of £1 is known, that of the like benefits, of any other amount, on the same status, is found by multiplying the known value by the number of pounds in that amount.

Lemma I.

(13.) It is required to determine the present value of a perpetual annuity of £1.

(14.) The present value of any perpetual annuity is evidently the sum of which that annuity is the annual interest, since this interest is produced annually without increase or diminution of the principal. Hence, if the interest of £1 for one year be denoted by r, £1 will be the present value of the perpetual annuity, r; and we shall have the following proportion: r is to £1, as the present value of the perpetual annuity, r, to the present value of the perpetual annuity, £1. That is,—

$$r : 1 :: 1 : \frac{1}{r}.$$

This, therefore, is the present value of a perpetuity, or perpetual annuity of £1.

(15.) It is usual to denote the present value of a perpetuity of £1 by P. So that when, as in the subsequent examples, and the tables hereto annexed, the rate of interest is 3 per cent., $P = 1 \div r = 1 \div {\cdot}03 = 33{\cdot}33333 +$.

(16.) P is for brevity called simply *the perpetuity*.

Lemma II.

(17.) It is required to determine the relation that subsists between the present value of an annuity on any given status, and that of an assurance on the same status.

(18.) Let the present value of the annuity be denoted by a, and that of the assurance by A. The annuity upon the given status, and a perpetuity due, to be entered upon at the end of the year in which that status fails, will evidently together compose a perpetuity commencing immediately. The present value of this perpetuity is P, that of the annuity is a, and the perpetuity due, *at the time of its being entered upon*, will be worth $1+P$. Hence, $P-a$ is the present value of an assurance of $1+P$ upon the given status. That is (12),—

$$P-a=A(1+P);$$

whence we obtain

$$A=\frac{P-a}{1+P};$$

and

$$a=P-(1+P)A.$$

(19.) *Cor.*—Adding unity to both sides of the last equation we obtain

$$1+a=1+P-(1+P)A=(1+P)(1-A);$$

and

$$1-A=\frac{1+a}{1+P}.$$

These two expressions we shall hereafter find useful.

PROBLEM I.

(20.) A perpetual annuity of £1 is now enjoyed by an individual of a given age. On the failure of the life at present in possession, a successor in the enjoyment of the annuity is to be nominated; and on the failure of this first succeeding life, a second, and so on for ever: subject to the conditions that each nomination is to be made at the end of the year in which the preceding life fails, and that all the lives shall, at the periods of their respective nominations, be of given ages. It is required to determine the present value of each succeeding life; or, the sum which ought now to be given for the right of nomination in each case.

(21.) In this and each of the following problems we are presented with two distinct cases. The first and more general case is, when the succeeding lives are to be at nomination of different ages; and the second, which is the more useful case, is when these lives are, at nomination, to be all of the same age. The two cases will be considered separately, after premising a few remarks as to the general principles on which the solution depends.

(22.) The annuity in the problem of course makes no cessation in its payments. Consequently, since the last payment received by each possessor is the payment which is made at the end of the year preceding that in which his death takes place, the first payment that is received by his successor is the payment which is made at the end of the year following that just mentioned. Hence it follows, that this being the period of the successor's nomination, the benefit to which each successor receives a title is an *annuity due* upon his life. And the value of this annuity, *at the time it is entered upon,* being known,* its *present* value will be the present value of an assurance of that amount on the status which consists of the successive duration of all the preceding lives, including that now in possession.

(23.) Thus, the present value of the first succeeding life will be that of an assurance on the life in possession, whose amount is the value of an annuity due, at the time of its being entered upon, on a life of the same age as the life in question is to be of at the time of his nomination. The present value of the second succeeding life will be that of an assurance on the status which consists of the successive duration of the life in possession and the first succeeding life, the amount of which assurance is the value, at the time of its being entered upon, of an annuity due on a life of the same age as the second succeeding life is to be of at the time of nomination; and so on.

(24.) It will be proper now to explain the notation we shall find it convenient to employ.

(25.) Annuities and assurances are denoted, the former by a, and the latter by A, with the age of the life on which they depend suffixed. Thus a_{30} and A_{30} denote, the former the present value of an annuity, and the latter the present value of an assurance, on a life now aged 30. But as it is necessary to possess the means of designating the order in which, when of different ages, the lives succeed each other, we shall strictly adhere to this notation only when the succeeding lives are to be at nomination all of the same age. So that, as in this case x will be employed to denote the age of the life in possession, and y that of the succeeding lives at the time of nomination, a_x and A_x will respectively denote the present value of an annuity and an assurance on the life in possession, and a_y and A_y the same respectively on a life now of the same age that the succeeding lives are to be of at the time of nomination.

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* By adding unity to the tabular value of an annuity on the given age, agreeably to (8).

(26.) But when the succeeding lives are to be of different ages at nomination, we shall employ as suffixes of a and A the numerals 0, 1, 2, 3, &c., of which the first (namely, 0) will belong to the life in possession, and the others to the succeeding lives in the order of their succession. The suffixes, when so used, will be distinguished by being inclosed in parentheses. We now return to the problem.

(27.) CASE 1.—When the succeeding lives are to be of different ages at nomination.

(28.) Keeping the preceding considerations in view, since $1+a_{(1)}$ is the value, at the time of its being entered upon, of an annuity due on the first succeeding life, the present value of an assurance of this amount on the life in possession, namely—

$$(1+a_{(1)})\mathrm{A}_{(0)}; \text{ or } (19),\ (1+\mathrm{P})(1-\mathrm{A}_{(1)})\mathrm{A}_{(0)},$$

will be the present value of the first succeeding life.

(29.) The value of the first life in succession is found thus readily because the preceding status consists of only a single life, and the values of assurances on such lives of all ages are given. As regards the second and following lives we have not the like facility. The values of assurances on the status which respectively precede them are not given. In each case, however, the previous results enable us to find the value of an annuity on the preceding status, from which, by means of the formula in (18), we can pass to that of an assurance.

(30.) The status which precedes the second life in succession, is the successive duration of the life in possession and the first succeeding life; and the value of an annuity upon that status is evidently the sum of the present values of annuities upon the lives which compose it. This sum is

$$a_{(0)}+(1+\mathrm{P})(1-\mathrm{A}_{(1)})\mathrm{A}_{(0)}.$$

Hence, substituting this for a in the formula

$$\frac{\mathrm{P}-a}{1+\mathrm{P}},$$

we have for the present value of an assurance of £1 on that status,

$$\frac{\mathrm{P}-a_{(0)}-(1+\mathrm{P})(1-\mathrm{A}_{(1)})\mathrm{A}_{(0)}}{1+\mathrm{P}}=\frac{\mathrm{P}-a_{(0)}}{1+\mathrm{P}}-(1-\mathrm{A}_{(1)})\mathrm{A}_{(0)}$$
$$=\mathrm{A}_{(0)}-(1-\mathrm{A}_{(1)})\mathrm{A}_{(0)}=\mathrm{A}_{(0)}\,.\,\mathrm{A}_{(1)};$$

and multiplying this by $1+a_{(2)}$, we have for the value of an assurance of $1+a_{(2)}$ on the same status (12), that is, for the value of the second life in succession,

$$(1+a_{(2)})A_{(0)}.A_{(1)};\ \text{or (19)},\ (1+P)(1-A_{(2)})A_{(0)}.A_{(1)}.$$

(31.) In like manner, the value of an annuity on the status which precedes the third life in succession is

$$\begin{aligned}&a_{(0)}+(1+P)(1-A_{(1)})A_{(0)}+(1+P)(1-A_{(2)})A_{(0)}.A_{(1)}\\ =&a_{(0)}+(1+P)[1-A_{(1)}+(1-A_{(2)})A_{(1)}]A_{(0)}\\ =&a_{(0)}+(1+P)(1-A_{(1)}.A_{(2)})A_{(0)};\end{aligned}$$

and substituting this, as before, for a in the formula

$$\frac{P-a}{1+P},$$

we find for the value of an assurance of £1 on this status,

$$\begin{aligned}&\frac{P-a_{(0)}-(1+P)(1-A_{(1)}.A_{(2)})A_{(0)}}{1+P}=\frac{P-a_{(0)}}{1+P}-(1-A_{(1)}.A_{(2)})A_{(0)}\\ &=A_{(0)}-(1-A_{(1)}.A_{(2)})A_{(0)}=A_{(0)}.A_{(1)}.A_{(2)}.\end{aligned}$$

Hence the value of an assurance of $1+a_{(3)}$ upon the same status, that is, of the third life in succession, will be

$$(1+a_{(3)})A_{(0)}.A_{(1)}.A_{(2)};\ \text{or},\ (1+P)(1-A_{(3)})A_{(0)}.A_{(1)}.A_{(2)}.$$

(32.) It is unnecessary to carry this operation further, since the law which the values follow is sufficiently developed. It is evidently such that, if we. denote by n any whole number whatsoever, the value of the nth life in succession will be represented by

$$(1+P)(1-A_{(n)})A_{(0)}.A_{(1)}.A_{(2)}\ldots.A_{(n-1)};$$

which therefore is the general solution of the problem proposed.

(33.) Case 2.—Suppose now that the succeeding lives are to be all of the same age at the periods of their respective nominations.

(34.) Representing, agreeably to (25), the age of the life in possession by x, and the common age of the succeeding lives at nomination by y, the value of the nth life in succession will be found by substituting, in the general expression just deduced, A_x for $A_{(0)}$, and A_y for each of the quantities $A_{(1)}$, $A_{(2)}$, $A_{(3)}$, &c. This evidently gives

$$(1+P)(1-A_y)A_x.A_y^{n-1};$$

which will denote the present value of the nth life when all the lives in succession are to be of the same age, y, at nomination.

(35.) In this expression, as in that from which it is derived, by giving to n any integer value whatsoever, we get the expression for the present value of the life corresponding to that value of n. Thus:—

If $n=1$, we have $(1+\mathrm{P})(1-\mathrm{A}_y)\mathrm{A}_x$,* for the value of the 1st life;
$n=2$, „ $(1+\mathrm{P})(1-\mathrm{A}_y)\mathrm{A}_x.\mathrm{A}_y$ „ „ 2nd life;
$n=3$, „ $(1+\mathrm{P})(1-\mathrm{A}_y)\mathrm{A}_x.\mathrm{A}_y^2$, „ „ 3rd life;

and so on.

PROBLEM II.

(36.) All things remaining as in Problem I., it is required to determine the total present value of the first n successive lives.

(37.) CASE 1.—*The succeeding lives of different ages.*—The value here required is evidently the sum of the n terms which arise by making n successively equal to 1, 2, 3, ... n, in the general expression of (32). Hence,

For $n=1$, we have $(1+\mathrm{P})(1-\mathrm{A}_{(1)})\mathrm{A}_{(0)}$;
$n=2$, „ $(1+\mathrm{P})(1-\mathrm{A}_{(2)})\mathrm{A}_{(0)}.\mathrm{A}_{(1)}$;
$n=3$, „ $(1+\mathrm{P})(1-\mathrm{A}_{(3)})\mathrm{A}_{(0)}.\mathrm{A}_{(1)}.\mathrm{A}_{(2)}$;
. . . .
. . . .
. . . .
$n=n$, „ $(1+\mathrm{P})(1-\mathrm{A}_{(n)})\mathrm{A}_{(0)}.\mathrm{A}_{(1)}.\mathrm{A}_{(2)}\ldots\mathrm{A}_{(n-1)}$.

Taking out the factor common to all these expressions, namely, $(1+\mathrm{P})\mathrm{A}_{(0)}$, multiplying out the remaining factors and adding the results (observing that one of the products which appears with the sign *plus* in each expression appears in the next with the sign *minus*), we obtain for the required value

$$(1+\mathrm{P})(1-\mathrm{A}_{(1)}.\mathrm{A}_{(2)}.\mathrm{A}_{(3)}\ldots\ldots\mathrm{A}_{(n)})\mathrm{A}_{(0)}.$$

(38.) *Cor.* 1.—If n be supposed infinitely great, then $\mathrm{A}_{(1)}$, $\mathrm{A}_{(2)}$, &c., being each less than unity, their product vanishes, and the expression—which in this case will denote the value of all the lives to perpetuity—is reduced to $(1+\mathrm{P})\mathrm{A}_{(0)}$; which, as we have already seen (12), is the present value of the assurance of a perpetuity due on the life in possession, and is altogether independent of the ages of the succeeding lives at nomination.

(39.) *Cor.* 2.—If from the value of all the lives just obtained we subtract that of the first n lives, we find for the value of all the lives after the nth—that is, of the $(n+1)$th and following lives—

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* For when $n=1$, A_y^{n-1} becomes A_y^0, which is equal to unity, whatever the value of A_y may be.

$$(1+\mathrm{P})\,\mathrm{A}_{(0)}\,.\,\mathrm{A}_{(1)}\,.\,\mathrm{A}_{(2)}\,\ldots\ldots\,\mathrm{A}_{(n)}.$$

Or, changing n into $n-1$, the value of the nth and following lives will be denoted by

$$(1+\mathrm{P})\,\mathrm{A}_{(0)}\,.\,\mathrm{A}_{(1)}\,.\,\mathrm{A}_{(2)}\,\ldots\ldots\,\mathrm{A}_{(n-1)}.$$

(40.) Case 2.—*The succeeding lives of the same age at nomination.*—In this case the expressions to be added are (34):—

$$\begin{array}{ll}
\text{For } n=1, & (1+\mathrm{P})(1-\mathrm{A}_y)\mathrm{A}_x \\
n=2, & (1+\mathrm{P})(1-\mathrm{A}_y)\mathrm{A}_x\,.\,\mathrm{A}_y \\
n=3, & (1+\mathrm{P})(1-\mathrm{A}_y)\mathrm{A}_x\,.\,\mathrm{A}_y^2 \\
\cdot & \cdot \quad \cdot \quad \cdot \\
\cdot & \cdot \quad \cdot \quad \cdot \\
\cdot & \cdot \quad \cdot \quad \cdot \\
n=n, & (1+\mathrm{P})(1-\mathrm{A}_y)\mathrm{A}_x\,.\,\mathrm{A}_y^{n-1}.
\end{array}$$

And their sum, which is the value required, is—

$$(1+\mathrm{P})(1-\mathrm{A}_y)\mathrm{A}_x(1+\mathrm{A}_y+\mathrm{A}_y^2+\mathrm{A}_y^3+\ldots\ldots+\mathrm{A}_y^{n-1})^*$$
$$=(1+\mathrm{P})(1-\mathrm{A}_y)\,\mathrm{A}_x\frac{1-\mathrm{A}_y^n}{1-\mathrm{A}_y}$$
$$=(1+\mathrm{P})(1-\mathrm{A}_y^n)\mathrm{A}_x.$$

(41.) The same result would have been obtained by making the requisite substitutions in the formula of Case 1; namely, A_x for $\mathrm{A}_{(0)}$, and A_y for each of the quantities $\mathrm{A}_{(1)}$, $\mathrm{A}_{(2)}$, &c.

(42.) *Cor.* 1.—If n be made infinite, A_y^n vanishes, and we have for the value of all the lives to perpetuity $(1+\mathrm{P})\mathrm{A}_x$, which corresponds with the expression already found (38).

(43.) *Cor.* 2.—Subtracting, as in (39), the value of the first n lives from this value of all the lives, and in the result changing n into $n-1$, we have for the value of the nth and following lives, to perpetuity,—

$$(1+\mathrm{P})\mathrm{A}_x\,.\,\mathrm{A}_y^{n-1}.$$

Problem III.

(44.) The same things remaining, it is required to determine the total present value of any n lives which follow each other at intervals of t lives; that is to say, of n lives which, commencing with the mth, include the $(m+t)$th, the $(m+2t)$th, the $(m+3t)$th, &c., and consequently terminate with the $(m+\overline{n-1}\,.\,t)$th; m, n, and t being any whole numbers whatsoever.

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* This is a decreasing series in geometrical progression, whose common ratio is A_y. The rule for summing such series is to be found in all elementary works on Algebra. It is as follows:—Multiply the last term by the common ratio; subtract the product from the first term; and divide the remainder by the excess of unity over the common ratio.

(45.) Case 1.—*The succeeding lives of different ages.*—The value here required will be the sum of the n terms which arise by the successive substitution of m, $m+t$, $m+2t$, ... $m+\overline{n-1}\,.\,t$, for n in the formula of (32). We thus have—

$$\begin{array}{ll}
\text{For } n=m, & (1+\mathrm{P})(1-\mathrm{A}_{(m)})\,\mathrm{A}_{(0)}\,.\,\mathrm{A}_{(1)}\,\ldots\ldots\,\mathrm{A}_{(m-1)}; \\
n=m+t, & (1+\mathrm{P})(1-\mathrm{A}_{(m+t)})\,\mathrm{A}_{(0)}\,.\,\mathrm{A}_{(1)}\,\ldots\,\mathrm{A}_{(m+t-1)}; \\
n=m+2t, & (1+\mathrm{P})(1-\mathrm{A}_{(m+2t)})\,\mathrm{A}_{(0)}\,.\,\mathrm{A}_{(1)}\,\ldots\,\mathrm{A}_{(m+2t-1)}; \\
\cdot\quad\cdot & \cdot\qquad\cdot\qquad\cdot\qquad\cdot \\
\cdot\quad\cdot & \cdot\qquad\cdot\qquad\cdot\qquad\cdot \\
\cdot\quad\cdot & \cdot\qquad\cdot\qquad\cdot\qquad\cdot \\
n=m+\overline{n-1}\,.\,t, & (1+\mathrm{P})(1-\mathrm{A}_{(m+\overline{n-1}\,.\,t)})\,\mathrm{A}_{(0)}\,.\,\mathrm{A}_{(1)}\,\ldots\ldots\,\mathrm{A}_{(m+\overline{n-1}\,.\,t-1)}.
\end{array}$$

And the sum of these is—

$$\begin{aligned}
(1+\mathrm{P})\,\mathrm{A}_{(0)}\,.\,\mathrm{A}_{(1)}\,\ldots\,\mathrm{A}_{(m-1)}\Big\{1-\mathrm{A}_{(m)}\Big[1+(1-\mathrm{A}_{(m+t)})\mathrm{A}_{(m+1)}\,.\,\mathrm{A}_{(m+2)}\,\ldots\,\mathrm{A}_{(m+t-1)} \\
+(1-\mathrm{A}_{(m+2t)})\,\mathrm{A}_{(m+1)}\,.\,\mathrm{A}_{(m+2)}\,\ldots\ldots\,\mathrm{A}_{(m+2t-1)}+\ldots \\
\ldots+(1-\mathrm{A}_{(m+\overline{n-1}\,.\,t)})\,\mathrm{A}_{(m+1)}\,.\,\mathrm{A}_{(m+2)}\,\ldots\ldots\,\mathrm{A}_{(m+\overline{n-1}\,.\,t-1)}\Big]\Big\};
\end{aligned}$$

which is the value required.

(46.) *Cor.* 1.—If $t=1$, the expression just found becomes—

$$\begin{aligned}
(1+\mathrm{P})\mathrm{A}_{(0)}\,.\,\mathrm{A}_{(1)}\ldots\mathrm{A}_{(m-1)}\Big\{1-\mathrm{A}_{(m)}\Big[1+(1-\mathrm{A}_{(m+1)})+(1-\mathrm{A}_{(m+2)})\mathrm{A}_{(m+1)} \\
+(1-\mathrm{A}_{(m+3)})\mathrm{A}_{(m+1)}\,.\,\mathrm{A}_{(m+2)}+\ldots+(1-\mathrm{A}_{(m+n-1)})\mathrm{A}_{(m+1)}\,.\,\mathrm{A}_{(m+2)}\ldots\mathrm{A}_{(m+n-2)}\Big]\Big\} \\
=(1+\mathrm{P})\,(1-\mathrm{A}_{(m)}\,.\,\mathrm{A}_{(m+1)}\,.\,\mathrm{A}_{(m+2)}\ldots\mathrm{A}_{(m+n-1)})\,\mathrm{A}_{(0)}\,.\,\mathrm{A}_{(1)}\ldots\ldots\mathrm{A}_{(m-1)};
\end{aligned}$$

which will denote the value of the mth and $n-1$ following lives; that is, of n lives, commencing with the mth.

(47.) *Cor.* 2.—And if m also be made equal to unity the expression last deduced becomes—

$$(1+\mathrm{P})\,(1-\mathrm{A}_{(1)}\,.\,\mathrm{A}_{(2)}\ldots\mathrm{A}_{(n)})\,\mathrm{A}_{(0)};$$

which is the expression already found (37) for the value of the *first* n lives.

(48.) Case 2.—*The succeeding lives of the same age at nomination.*—In this case the expressions to be added are those that arise from the successive substitution of m, $m+t$, $m+2t$, ... $m+\overline{n-1}\,.\,t$, for n, in the formula of (40). We thus have :—

$$\begin{array}{ll}
\text{For } n=m, & (1+\mathrm{P})(1-\mathrm{A}_y)\mathrm{A}_x.\mathrm{A}_y^{m-1} \\
n=m+t, & (1+\mathrm{P})(1-\mathrm{A}_y)\mathrm{A}_x.\mathrm{A}_y^{m+t-1} \\
n=m+2t, & (1+\mathrm{P})(1-\mathrm{A}_y)\mathrm{A}_x.\mathrm{A}_y^{m+2t-1} \\
\cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\
n=m+\overline{n-1}.t, & (1+\mathrm{P})(1-\mathrm{A}_y)\mathrm{A}_x.\mathrm{A}_y^{m+\overline{n-1}.t-1}.
\end{array}$$

And the sum of these expressions is—

$$(1+\mathrm{P})(1-\mathrm{A}_y)\mathrm{A}_x.\mathrm{A}_y^{m-1}(1+\mathrm{A}_y^{t}+\mathrm{A}_y^{2t}+\mathrm{A}_y^{3t}+\ldots+\mathrm{A}_y^{\overline{n-1}.t})$$
$$=\frac{(1+\mathrm{P})(1-\mathrm{A}_y)(1-\mathrm{A}_y^{nt})\mathrm{A}_x.\mathrm{A}_y^{m-1}}{1-\mathrm{A}_y^{t}};$$

which will denote the present value required.

(49.) *Cor.* 1.—If $t=1$, that is, if the lives in question immediately follow each other, we have for the value of n lives, commencing with the mth,

$$(1+\mathrm{P})(1-\mathrm{A}_y^{n})\mathrm{A}_x.\mathrm{A}_y^{m-1}.$$

(50.) *Cor.* 2.—And if also $m=1$, we have from the last expression, for the value of the first n lives—

$$(1+\mathrm{P})(1-\mathrm{A}_y^{n})\mathrm{A}_x,$$

as already found (40).

(51.) *Cor.* 3.—If in the expression deduced in (48) we suppose n infinitely great, A_y^{nt} will vanish, and we shall then have for the value of the mth and every tth following life for ever—

$$\frac{(1+\mathrm{P})(1-\mathrm{A}_y)\mathrm{A}_x.\mathrm{A}_y^{m-1}}{1-\mathrm{A}_y^{t}}.$$

(52.) *Cor.* 4.—Making in this expression $t=1$, we have for the value of the mth and following lives for ever—

$$(1+\mathrm{P})\mathrm{A}_x.\mathrm{A}_y^{m-1};$$

which has been already found (43).

Scholium.

(53.) In several of the corollaries to the present and foregoing Problems previous results are reproduced, thus showing, to a certain extent at least, that the different expressions which yield the same results are consistent with each other. As, however, the results thus obtained are not new results, the matter of the corollaries referred to might, with perhaps equal propriety, have been reserved for this place, in which it is proposed still further to show the consistency of previous results with each other.

(54.) Resuming the expression deduced in (48), namely—

$$\frac{(1+\mathrm{P})\,(1-\mathrm{A}_y)\,(1-\mathrm{A}_y^{nt})\,\mathrm{A}_x\,.\,\mathrm{A}_y^{m-1}}{1-\mathrm{A}_y^{t}},$$

which denotes the present value of n lives, commencing with the mth, and following each other at intervals of t lives: it is obvious that, so long as t and n do not vary, the n lives, whose value is indicated by the formula, will depend altogether upon the value of m, and that for every different value of m, less than $t+1$, it will be an entirely different set of lives whose value is thus indicated. From this it further appears, that if 1, 2, 3, &c., up to t, be successively written for m, and the results added, the sum will be the value of the first tn lives, since each expression will denote the value of n lives, and the number of expressions will be t.

(55.) Now, by changing n into tn in the formula of (40), we know already that the value of the first tn lives is

$$(1+\mathrm{P})\,(1-\mathrm{A}_y^{tn})\,\mathrm{A}_x.$$

The coincidence with this expression, therefore, of that to be deduced as above, will establish the consistency with each other of the two from which they are respectively derived.

(56.) The expression of (48), quoted above, becomes—

$$\begin{array}{ll}
\text{When } m=1, & \dfrac{(1+\mathrm{P})\,(1-\mathrm{A}_y)\,(1-\mathrm{A}_y^{nt})\,\mathrm{A}_x}{1-\mathrm{A}_y^{t}} \\[2ex]
m=2, & \dfrac{(1+\mathrm{P})\,(1-\mathrm{A}_y)\,(1-\mathrm{A}_y^{nt})\,\mathrm{A}_x\,.\,\mathrm{A}_y}{1-\mathrm{A}_y^{t}} \\[2ex]
m=3, & \dfrac{(1+\mathrm{P})\,(1-\mathrm{A}_y)\,(1-\mathrm{A}_y^{nt})\,\mathrm{A}_x\,.\,\mathrm{A}_y^{2}}{1-\mathrm{A}_y^{t}} \\
\cdot\ \cdot & \cdot \\
\cdot\ \cdot & \cdot \\
\cdot\ \cdot & \cdot \\
m=t, & \dfrac{(1+\mathrm{P})\,(1-\mathrm{A}_y)\,(1-\mathrm{A}_y^{nt})\,\mathrm{A}_x\,.\,\mathrm{A}_y^{t-1}}{1-\mathrm{A}_y^{t}};
\end{array}$$

and the sum of these is—

$$\frac{(1+\mathrm{P})\,(1-\mathrm{A}_y)\,(1-\mathrm{A}_y^{nt})\,\mathrm{A}_x}{1-\mathrm{A}_y^{t}}(1+\mathrm{A}_y+\mathrm{A}_y^{2}+\ldots+\mathrm{A}_y^{t-1})$$

$$=\frac{(1+\mathrm{P})\,(1-\mathrm{A}_y)\,(1-\mathrm{A}_y^{nt})\,\mathrm{A}_x}{1-\mathrm{A}_y^{t}}\cdot\frac{1-\mathrm{A}_y^{t}}{1-\mathrm{A}_y}$$

$$=(1+\mathrm{P})\,(1-\mathrm{A}_y^{nt})\,\mathrm{A}\ ;$$

by which the theorem is established.

(57.) By applying a similar process to the formula of (51), which denotes the value of the *m*th and every *t*th following life to perpetuity, we should obtain as the value of all the lives to perpetuity, $(1+\mathrm{P})\mathrm{A}_x$, agreeing with the expression in (38).

PROBLEM IV.

(58.) All things continuing as in the previous Problems, a fine of £1 is to be paid at the end of the year in which each life in possession fails; that is, at the time of the nomination of the next succeeding life. It is required to determine the present value of each fine.

(59.) CASE 1.—*The succeeding lives of different ages at nomination.*—The first fine is paid on the extinction of the life in possession, the second on the extinction of the first succeeding life, and, generally, the *n*th fine is paid on the extinction of the $(n-1)$th succeeding life. Now the present value of the first fine is evidently A_0; and by the investigation of the first Problem the present values of the 2nd and 3rd fines were found to be $\mathrm{A}_{(0)}.\mathrm{A}_{(1)}$, and $\mathrm{A}_{(0)}.\mathrm{A}_{(1)}.\mathrm{A}_{(2)}$, respectively. It thus appears that the law which the values follow is such that the value of the *n*th fine will be denoted by—

$$\mathrm{A}_{(0)}.\mathrm{A}_{(1)}.\mathrm{A}_{(2)}.\mathrm{A}_{(3)}\ldots.\mathrm{A}_{(n-1)}.$$

(60.) The same thing may be also made to appear thus. The value of the *n*th fine is that of an assurance of £1 on the status which terminates with the failure of the $(n-1)$th succeeding life. Now the value of the *n*th life, namely, $(1+\mathrm{P})\,(1-\mathrm{A}_n)\,\mathrm{A}_{(0)}.\mathrm{A}_{(1)}\ldots\ldots\mathrm{A}_{(n-1)})$, (32,) being the value of an assurance of $(1+a_{(n)})$ on the same status (22), it is evident that by dividing the value of the *n*th life by $1+a_{(n)}$, or its equivalent (18), $(1+\mathrm{P})\,(1-\mathrm{A}_{(n)})$, we should obtain the value of the *n*th fine. The result of this division is manifestly

$$\mathrm{A}_{(0)}.\mathrm{A}_{(1)}.\mathrm{A}_{(2)}\ldots.\mathrm{A}_{(n-1)},$$

as already found.

(61.) CASE 2.—*The succeeding lives of the same age at nomination.*—The value of the *n*th fine in this case is deduced from that in the last, by substituting, in the expression there found, A_x for $\mathrm{A}_{(0)}$, and A_y for each of the quantities $\mathrm{A}_{(1)}$, $\mathrm{A}_{(2)}$, &c. This substitution gives

$$\mathrm{A}_x.\mathrm{A}_y^{n-1}$$

for the value required.

(62.) By giving to n in these expressions the requisite value, the value of any fine that may be proposed (that is, whose place in the order of lives is assigned) will be found.

Problem V.

(63.) The same things remaining, it is required to determine the present value of the first n fines.

(64.) Case 1.—*The succeeding lives of different ages at nomination.*—The value here required will be the sum of the n terms which arise by the successive substitution of 1, 2, 3, ... n for n in the formula of (59). Making these substitutions, we have—

$$\begin{array}{ll} \text{For } n=1, & A_{(0)}; \\ n=2, & A_{(0)}.A_{(1)}; \\ n=3, & A_{(0)}.A_{(1)}.A_{(2)}; \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ n=n, & A_{(0)}.A_{(1)}.A_{(2)}\ldots\ldots A_{(n-1)}. \end{array}$$

And the sum of these expressions, which is the required value, is—

$$A_{(0)}+A_{(0)}.A_{(1)}+A_{(0)}.A_{(1)}.A_{(2)}+\ldots+A_{(0)}.A_{(1)}.A_{(2)}\ldots\ldots A_{(n-1)}.$$

Here each term can be readily deduced from that which precedes it; but no expression can be assigned for the sum of the terms which would materially simplify the arithmetical computation.

(65.) Case 2.—*The succeeding lives of the same age at nomination.*—The value here required is evidently the sum of the series:—

$$\begin{aligned} & A_x+A_x.A_y+A_x.A_y^2+A_x.A_y^3+\ldots+A_x.A_y^{n-1} \\ & =A_x(1+A_y+A_y^2+A_y^3+\ldots+A_y^{n-1}) \\ & =\frac{A_x(1-A_y^n)}{1-A_y}. \end{aligned}$$

(66.) *Cor.* 1.—If the value of all the fines to perpetuity be required, we have for this value, by making n in the expression last deduced infinite, in consequence of which A_y^n vanishes—

$$\frac{A_x}{1-A_y}.$$

(67.) *Scholium.*—This expression is not, like the analogous one in (38), for the value of all the successive lives to perpetuity, independent of y, the age of the nominees at appointment. It shows that, since after the earlier years of existence the value of an assurance generally increases with the age of the life on which that assurance depends, the older the age of nomination is, the greater will be the value of the fines to the receiver; which is indeed sufficiently evident without reference to the formula.

(68.) *Cor.* 2.—By subtracting from the value of all the fines just found, that of the first n fines, we obtain for the value of all the fines after the nth, that is, of the $(n+1)$th and following fines, to perpetuity—

$$\frac{A_x - A_x(1-A_y^n)}{1-A_y} = \frac{A_x . A_y^n}{1-A_y}.$$

And this expression, by changing n into $n-1$, becomes—

$$\frac{A_x . A_y^{n-1}}{1-A_y};$$

which will denote the value of the nth, and following fines, for ever.

⁂ Part II., containing the Praxis, will appear in a subsequent Number.

Sketch of the recent Progress of the Assurance of Life and Property on the Continent. By Samuel Brown, Esq., *Member of the Council and one of the Honorary Secretaries of the Institute of Actuaries, and Actuary of the Mutual Life Assurance Society.*

[Continued from vol. i., page 293.]

Life Assurance in France.

THE first association in France of the nature of Life Assurance is to be attributed to Laurent Tontini, an Italian banker, who settled in Paris in 1653. In 1681 these institutions were prohibited by Royal Ordinance, as being illegal and contrary to public morality, on the ground that no value should be set on the life of a freeman, which is beyond all price. A new Tontine was established in 1759, but was suppressed by Letters Patent in 1770, to reappear in 1791 under the name of *Caisse Lafarge.*

In 1787, however, a Company had been formed for the purpose of Life Assurance, on a similar plan to the English Companies, under the name of the Compagnie Royale d'Assurance. The Decree of the Conseil d'Etat authorized its perpetuity with a pri-